

MOTIVES OF UNITARY AND ORTHOGONAL HOMOGENEOUS VARIETIES

DANIEL KRASHEN

ABSTRACT. Grothendieck-Chow motives of quadric hypersurfaces have provided many insights into the theory of quadratic forms. Subsequently, the landscape of motives of more general projective homogeneous varieties has begun to emerge. In particular, there have been many results which relate the motive of a one homogeneous variety to motives of other simpler or smaller ones (see for example [Kar00, CGM05, Bro05, NSZ, NK, BVNK, Zai]). In this paper, we exhibit a relationship between motives of two homogeneous varieties by producing a natural rational map between them which becomes a projective bundle morphism after being resolved. This allows one to use formulas for projective bundles and blowing up (as in [Man68]) to relate the motives of the two varieties. We believe that in the future this ideas could be used to discover more relationships between other types of homogeneous varieties.

1. INTRODUCTION

Let F be a field of characteristic not 2, and let L/F be a quadratic extension with Galois group $G = \langle \sigma \rangle$. Let W be an L -vector space and h an L/F Hermitian form. We will abbreviate this by saying that (W, h) is an L/F Hermitian space.

Both quadratic and Hermitian spaces give rise to projective homogeneous varieties as follows. For a quadratic space (W, q) , we let $V(q)$ denote the associated quadric hypersurface in $\mathbb{P}(W)$ defined by the vanishing of q . This is a homogeneous space for the orthogonal group $O(W, q)$.

If (W, h) is an L/F -Hermitian space we denote by $V(h)$ the variety of isotropic L -lines in W . This is an F -variety and may be regarded as a closed subvariety of $tr_{L/F}\mathbb{P}_L(W)$ which in turn is a closed subvariety of the Grassmannian $Gr(2, W)$ of 2-dimensional F -subspaces of W . It is a homogeneous space for the unitary group $U(W, h)$, and is a twisted form of the variety of flags consisting of a dimension 1 and codimension 1 linear subspace in a n dimensional vector space (where $\dim_F(W) = 2n$).

For a regular F -scheme X , we let $M(X)$ denote the Grothendieck-Chow motive of X as described in [Man68]. For a motive M , we let $M(i)$ denote the i 'th Tate twist of M . Our main result is the following:

Theorem. 3.3. *Let (W, h) be an L/F -Hermitian space, and let q_h be the associated quadratic form on W . Then we have an isomorphism of motives*

$$M(V(q_h)) \oplus \bigoplus_{i=1}^{n-2} M(\mathbb{P}_L(N))(i) \cong M(V(h)) \oplus M(V(h))(1)$$

where $\dim_F(N) = 2n$.

Comparing the dimension 0 Chow groups of each side, we obtain:

Corollary. 3.4.

$$A_0(V(h)) \cong A_0(V(q_h)) = \begin{cases} \mathbb{Z} & \text{if } q \text{ is isotropic} \\ 2\mathbb{Z} & \text{if } q \text{ is anisotropic,} \end{cases}$$

in particular, the degree map $A_0(V_h) \rightarrow \mathbb{Z}$ is injective.

2. PROJECTIVE SPACES

Definition 2.1. *Let L be an étale F -algebra, and let N be an L -module of rank $m < \infty$. We define $\mathbb{P}(N, L)$ to be $\mathrm{tr}_{L/F}\mathbb{P}_L(N)$ which may be regarded as a closed subvariety of $\mathrm{Gr}(\dim(L), N)$, where N is considered here as an F -space.*

Definition 2.2. $S(N) = \{[m] \in \mathbb{P}_F(N) \mid \mathrm{ann}_L(m) \neq 0\}$.

We note the following easy lemma:

Lemma 2.3. *Let L be a commutative étale F -algebra, and let N be a finite dimensional free L -module. Then we have a rational morphism $f = f_N : \mathbb{P}_F(N) \dashrightarrow \mathbb{P}(N, L)$ defined by taking a 1-dimensional linear subspace $l \subset N$ to its (usually $\dim(L)$ -dimensional) L -span, Ll . This is well defined on the open complement of $S(N)$ in $\mathbb{P}_F(N)$.*

This result may be proved by descent. We omit the proof, but note that it is essentially contained in the proof of the following result.

Lemma 2.4. *Suppose L is either a quadratic étale extension of F . Let N be a finite rank free L -module, and let $S(N) = \{[m] \in \mathbb{P}_F(N) \mid \mathrm{ann}_L(m) \neq 0\}$. Then there is a well defined morphism $g_N : \mathrm{Bl}_{S(N)}\mathbb{P}_F(N) \rightarrow \mathbb{P}_L(N)$ extending the rational map f_N . Furthermore, g_N is a projective bundle of rank 1.*

Proof. By descent, it suffices to consider the case where F is algebraically closed. In this case, we have $L = Fe_1 \oplus Fe_2$ and so $N = N_1 \oplus N_2$, where each $N_i = Ne_i$ is a vector space of finite dimension n . This gives

$$\mathbb{P}_L(N) = \mathbb{P}_F(N_1) \times \mathbb{P}_F(N_2),$$

and

$$S(N) = \{[n_1; n_2] \in \mathbb{P}_F(N) \mid \text{either } n_i \text{ or } n_j \text{ is } 0\},$$

where the rational morphisms $\mathbb{P}_F(N) \rightarrow \mathbb{P}_F(N_i)$ are given by projections. One may check directly that the desired morphism g_N is just the resolution of the rational morphism. More precisely, we have an isomorphism

$$Bl_{S(N)}\mathbb{P}_F(N) = \left\{ ([n_1; n_2], [m_1], [m_2]) \in \mathbb{P}_F(N) \times \mathbb{P}_F(N_1) \times \mathbb{P}_F(N_2) \mid \begin{array}{l} n_i \text{ is in} \\ \text{the line } [m_i] \end{array} \right\}.$$

In particular, this is a \mathbb{P}^1 bundle as the inverse image of $[n_1], [n_2]$ is given by $\{[\lambda n_1; \mu n_2] \mid \lambda, \mu \in F\}$, which is isomorphic as a variety with a Galois action to $\mathbb{P}_F(L)$. \square

Lemma 2.5. *Let L be a quadratic étale extension of F . Using the notation of lemma 2.4, there is an isomorphism $S(N) \cong \mathbb{P}_L(N)$.*

Proof. Choose an L basis n_1, \dots, n_k for N . Since L is étale, we may write $L_{\overline{F}} = \overline{F}e_1 \oplus \overline{F}e_2$, where \overline{F} is the separable closure of F . Then $S(N)_{\overline{F}}$ is the disjoint union of the schemes P_i , $i = 1, 2$, where $P_i = \mathbb{P}_{\overline{F}}(W_i)$ and W_i is the \overline{F} -span of the vectors $n_j e_i$ as j varies. Examining the G_F -Galois action, we see that we have an isomorphism of \overline{F} -schemes between $S(N)_{\overline{F}}$ and $\mathbb{P}_{\overline{F}}(W) \times_{\overline{F}} \text{Spec}(L_{\overline{F}})$, where W is a k -dimensional \overline{F} vector space. By descent, this gives $S(N) = \mathbb{P}_L(V)$, where V is a rank k free L -module, and of course we may choose $V = N$. \square

As a result of this we obtain information on the motive $M(\mathbb{P}_L(N))$:

Theorem 2.6. *Let L/F be a quadratic étale extension. Then we have an isomorphism in the category of Grothendieck-Chow motives:*

$$\begin{aligned} M(\mathbb{P}_F(N)) \oplus \bigoplus_{i=1}^{n-1} M(\mathbb{P}_L(N))(i) &\cong M(\mathbb{P}(N, L)) \oplus M(\mathbb{P}(N, L))(1) \\ &\cong M(\text{tr}_{L/F}\mathbb{P}(N)) \oplus M(\text{tr}_{L/F}\mathbb{P}(N))(1) \end{aligned}$$

where $\dim_F(N) = 2n$.

Proof. This follows from the formulas in [Man68] for the motive of a projective bundle and for the motive of a blow up. \square

3. QUADRATIC AND HERMITIAN SPACES

Let (W, h) be an L/F Hermitian space. Write h in diagonal form as $h = \langle a_1, \dots, a_n \rangle$, taken with respect to a basis w_1, \dots, w_n for W over L . If $L = F(\beta)$ with $\beta^2 = b \in F$, we may consider the basis $u_i = w_i, v_i = \beta w_i$ for W as an F -space. Note that we may define a quadratic form q_h on W as an F -space by $v \mapsto q_h(v) = h(v, v)$. In this basis this is exactly the form $\langle 1, -b \rangle \otimes \langle a_1, \dots, a_n \rangle$.

We note the following observation:

Lemma 3.1. *Let (W, h) be an L/F Hermitian space. Then $w \in W$ is isotropic for h if and only if it is isotropic for q_h .*

Consequently, if we consider the map from the previous section:

$$f = f_W : \mathbb{P}_F(W) \dashrightarrow \mathbb{P}(W, L),$$

we obtain by restriction a rational map

$$\phi : V(q_h) \dashrightarrow V(h).$$

Lemma 3.2. *Suppose L is either a quadratic étale extension of F . Let W be a finite dimensional free L -module, and let*

$$S(W) = \{[m] \in \mathbb{P}_F(W) \mid \text{ann}_L(m) \neq 0\}.$$

Then $S(W) \subset V(q_h)$ and the morphism $g_W : \text{Bl}_{S(W)} \mathbb{P}_F(W) \rightarrow \mathbb{P}_L(W)$ from lemma 2.4 restricts to a morphism $g_h : \text{Bl}_{S(W)} V(q_h) \rightarrow V(h)$.

Proof. Suppose that $S(W) \subset V(q_h)$. Then by the standard properties of the blow up ([Ful98], B.6.10), we get a natural (inclusion) morphism $\text{Bl}_{S(W)} V(q_h) \rightarrow \text{Bl}_{S(W)} V(h)$, and so by restriction a morphism $\text{Bl}_{S(W)} V(q_h) \rightarrow \mathbb{P}_L(W)$. Since this map agrees with ϕ on a dense open set, it follows that the image is exactly $V(h) \subset \mathbb{P}_L(W)$.

It remains to show that $S(W) \subset V(q_h)$. To show this, it suffices to assume that $\overline{F} = F$, and so $L \cong F \oplus F$, say with idempotent basis e_1, e_2 . In this case we may suppose that the Galois involution on L is σ defined by $\sigma(e_1) = e_2, \sigma(e_2) = e_1$, and we may assume $h = \langle 1, \dots, 1 \rangle$ with respect to a given basis w_i . Choosing the basis $u_i = e_1 w_i, v_i = e_2 w_i$ for W as an F -space, we see that for a vector

$$w = \sum_{i=1}^n \lambda_i u_i + \mu_i v_i,$$

$w \in S(W)$ if and only if either $\lambda_i = 0$ all i or $\mu_i = 0$ all i . Since

$$q_h(w) = \sum \lambda_i \mu_i,$$

it immediately follows that $S(W) \subset V(q_h)$ as desired. \square

Since the morphism g_h is obtained by restriction of g_W , it is also a projective bundle morphism of relative dimension 1, and again we have from the motivic blow-up and projective bundle formulas ([Man68]):

Theorem 3.3. *Let (W, h) be an L/F -Hermitian space, and let q_h be the associated quadratic form on W . Then we have an isomorphism of motives*

$$M(V(q_h)) \oplus \bigoplus_{i=1}^{n-2} M(\mathbb{P}_L(W))(i) \cong M(V(h)) \oplus M(V(h))(1)$$

where $\dim_F(W) = 2n$.

In particular, comparing the dimension 0 Chow groups of each side, we obtain:

Corollary 3.4.

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